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Dispersion Mapping Theorems

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1. INTRODUCTION

Many familiar mathematical questions can be restated in the following form: "When is A more complex than B, and how do you put the answer in quantitative form?" This has been answered in a variety of ways, depending on the category to which A and B belong. Two recursive functions have been compared by their index numbers in any universal listing (Kolmogorov: see [7]). In classical analysis, a function of three real variables seems more complicated than a function of only two, while a function with continuous fourth derivatives seems simpler than one that is merely continuous. Vitushkin discovered that the index n/p is a useful measure of the complexity of the entire class of functions of n real variables having continuous pth derivatives [4, 5]. However, this approach is not appropriate if one is dealing with functions that are merely continuous ($p = 0$), or when one is dealing with individual functions and not classes. Moreover, one would like to use the term "simple" for functions that can be approximated arbitrarily well by simple functions, even though they themselves are not "simple."

Nor is it enough merely to count the number of variables. A function of the form

$$F(x, y, z) = f(g(x, y), h(y, z)) \tag{1}$$

is a function of three real variables, but since it is built from functions of two variables, it ought to be quantitatively simpler than the general continuous function of three variables.

Several years ago, I observed that such questions can also be stated in terms of mapping diagrams (see [1]). For example, consider those functions of five variables that can be represented in the format

$$F(x, y, z, u, v) = f(\Phi(x, y, z), u, v) \tag{2}$$

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in terms of continuous functions of only three variables. First suppress (u, v) by introducing $Z = C[R^2]$, and writing $F(x, y, z, u, v)$ as $F(x, y, z)(u, v)$, so that F is now seen as a function from R^3 to Z. Then, (2) requires that $F = f \circ \Phi$, where Φ is a continuous function from R^3 to R and f is a continuous function from R to Z. Thus, (2) asks us to examine those maps F from R^3 to Z which can be factored through R, as shown in the diagram below:

$$\begin{array}{ccc} R^3 & \xrightarrow{\Phi} & R \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle f} \\ & & Z \end{array} .$$

Other examples can be treated in a similar way. For example, to examine (1), first introduce special maps Φ from R^4 to R^2 of the form $\Phi(t_1, t_2, t_3, t_4) = (g(t_1, t_2), h(t_3, t_4))$:

$$\begin{array}{ccccc} R^2 & \times & R^2 & = & R^4 \\ {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Phi} \\ R & \times & R & = & R^2 \end{array} .$$

Then construct $X \subset R^4$, homeomorphic to R^3, by the special embedding $(x, y, z) \to (x, y, y, z)$. Then, the class of mappings F with the special representation (1) can be regarded as those maps of X to R which factor through R^2 by one of the special maps Φ, as shown below:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & R^2 \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle f} \\ & & R \end{array} .$$

Examination of these suggests that one study a general factoring problem. Choose spaces X, Y, and Z, and then within the class $C[X, Z]$ of all continuous mappings F from X into Z we identify the subclass $\mathscr{F}_Y$ of those F that can be factored through Y, $F = f \circ \Phi$, regarding these as "simple."

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle f} \\ & & Z \end{array} .$$

The objective is to find properties that are characteristic of the maps in $\mathscr{F}_Y$ and of those mappings that can be approximated uniformly by mappings in

$\mathscr{F}_Y$. In particular, we would like to know conditions on X, Y, and Z that guarantee that $\mathscr{F}_Y$ is a small subset of $C[X, Z]$, and find quantitative estimates for the size of $\mathscr{F}_Y$.

The results in the present paper are only a beginning, intended to show that useful theorems can be obtained in a number of cases related directly to problems dealing with the approximate complexity of functions. The approach is via Kolmogorov ε-entropy and entropy dimension, and some of the results obtained in Section 3 on dispersion functions may have wider usefulness in the study of continuous mappings; the combinatorial lemma on colored graphs may also be of interest, as may be the observations on dimension increasing maps in Section 5.

2. Heuristic Arguments

Let X and Y be metric spaces and $C[X, Y]$ the space of all continuous maps from X into Y, with the uniform metric. We regard Y as simpler than X if Y can be faithfully embedded in X but not conversely. In this case, any $\Phi \in C[X, Y]$ must fail to be 1-to-1, and must therefore compress some complex aspect of X. If $c(S)$ is an appropriate quantitative measure of the complexity of subsets S of X, then since

$$X = \bigcup_{y \in Y} \Phi^{-1} y$$

we are led to hope that

$$c(X) = \{c(Y)\} \times \{\text{average value of } c(\Phi^{-1} y)\}$$

and dividing by $c(Y)$, that

$$\max_{y \in Y} c(\Phi^{-1} y) \geqslant \frac{c(X)}{c(Y)}. \tag{3}$$

Note that the right side is independent of Φ.

This heuristic reasoning has led to the conjecture that when Y is simpler than X, every admissible mapping Φ from X into Y must have at least one point-inverse $\Phi^{-1} y$ which achieves at least a certain minimal complexity, independent of Φ.

Results of this type already exist in the literature. If $c(S) = 2^{\dim(S)}$, where $\dim(S)$ is the classical topological dimension of S, then (3) holds since it is equivalent to the assertion that any continuous map from a space X into a space Y of smaller (finite) dimension must have a point-inverse of dimension $\dim(X) - \dim(Y)$ [3].

For our purposes we need analogues of this, using a measure $c(S)$ related to Kolmogorov entropy. If S is an infinite subset of a compact metric space, and $\delta > 0$, then a δ-dispersed subset of S is a set $x_1, x_2, \dots, x_m$ such that $d(x_i, x_j) \geqslant \delta$ for all $i \neq j$. Then, for $c(S)$ we will use

$$N(S, \delta) = \text{the maximum number of points in a } \delta\text{-dispersed subset of } S. \tag{4}$$

The rate of increase of $N(S, \delta)$, as δ decreases, describes the size or capacity of S. If $N(S, \delta) \approx C\delta^{-p}$ as $\delta \downarrow 0$, we say that S has entropy dimension p. An n-cell has entropy dimension n. Sets in R^n with fractional entropy dimension are easily constructed, and numerous examples can be seen in the fascinating book by Mandelbrot [6].

If X and Y are metric spaces, with X compact, and $\Phi \in C[X, Y]$, set

$$M(\Phi, \delta) = \max_{y \in Y} N(\Phi^{-1}y, \delta). \tag{5}$$

This is integer valued, and is the maximum δ dispersion of any point-inverse of Φ. Finally, with the conjecture (3) in mind, we say that $C[X, Y]$ admits a dispersion function $K(\delta)$ if it is true that $K(\delta)$ is an unbounded increasing function of δ such that for all sufficiently small δ and any mapping Φ in $C[X, Y]$, $M(\Phi, \delta) \geqslant K(\delta)$. In the next section, we show that certain classes of mappings admit dispersion functions; we conjecture that this is always the case when Y is simpler than X.

3. Dispersion Mapping Theorems

In this section, we obtain dispersion functions for the class of real valued functions on a p-cell.

Let A and B be compact metric spaces, each arcwise connected, and let $X = A \times B$, with the metric

$$d(x_1, x_2) = d(a_1, a_2) + d(b_1, b_2).$$

Theorem 1. *A dispersion function for $C[X, R]$ is given by*

$$K(\delta) = \textit{the smaller of } N(A, \delta), N(B, \delta). \tag{6}$$

Proof. Let $\Phi \in C[X, R]$. For any $a \in A$, let $B_a = \{a\} \times B$, and consider the image sets $\Phi(B_a)$. Since B is connected, each is an interval I_a of reals. If the intersection of all these intervals is nonempty, choose a real number v in the intersection and set $S = \Phi^{-1}(v)$. Then S is a subset of X that meets each of the sets B_a. Given $\delta > 0$, let $N = N(A, \delta)$ and choose points $a_k \in A$, $k =$

$1, 2, \ldots, N$, that are δ-dispersed, and then $b_k \in B$ so that $x_k = (a_k, b_k) \in S$. These form a δ-dispersed set of N points of S, and we have shown that

$$M(\Phi, \delta) \geqslant N(S, \delta) \geqslant N = N(A, \delta).$$

Suppose now that $\bigcap I_a$ is empty. Choose a' and a'' in A so that $I_{a'}$ and $I_{a''}$ are disjoint, and a real number v lying between these intervals. Again, set $S = \Phi^{-1}(v)$; any arc in X joining a point of $B_{a'}$ and a point of $B_{a''}$ must intersect S. Given $\delta > 0$, let $N = N(B, \delta)$ and choose N points b_k in B, δ-dispersed. Let β be an arc in A with end points a', a'' and let β_k be the arc $\beta \times \{b_k\}$ in X, connecting $B_{a'}$ and $B_{a''}$, and x_k a point of β_k in S. Since β_i and β_j are everywhere δ apart in X, the N points x_k are δ-dispersed in S and $M(\Phi, \delta) \geqslant N(S, \delta) \geqslant N = N(B, \delta)$, completing (6).

For a k-cell, we have $N(I^k, 1) = 2^k$, $N(I^k, 1/m) = (m+1)^k$, and in general, $N(I^k, \delta) \geqslant \delta^{-k}$. If we factor a p-cell X as $I^p = A \times B$, where A and B are cells of dimension $[p/2]$ and $p - [p/2]$, Theorem 1 gives us:

COROLLARY. *If $I = [0, 1]$, the class $C[I^p, R]$ has a dispersion function $K(\delta)$ with $K(1/m) = (m+1)^{[p/2]}$ and, as $\delta \downarrow 0$, $K(\delta) \geqslant \delta^{-[p/2]}$.*

This estimate is not best possible. The heuristic argument in Section 2 suggests that a correct value ought to be $K(\delta) \approx N(I^p, \delta)/N(I, \delta) \approx \delta^{-(p-1)}$. We verify this next.

THEOREM 2. *A dispersion function for $C[I^p, R]$ is given by $K(\delta) = [(C/\delta)^{p-1}]$, where*

$$C = \tfrac{1}{3} 2^{-p/2}. \tag{7}$$

We reduce the proof of this to a combinatorial problem on colored graphs, which in turn is proved by induction on p. The elementary proof given below was discovered after I had seen an elegant but much more complicated argument by Andreas Blass, which also produced a far smaller value for C.

Proof. Let n be an integer larger than 15 and $\delta = 1/(n-1)$. In the p-cell I^p, construct the regular rectangular lattice of vertices P_k spaced evenly with separation δ; if k is the multi-index $(k_1, k_2, \ldots, k_p)$, with $0 \leqslant k_j \leqslant p$, then $P_k = \delta k$. Let $\Phi \in C[X, R]$ and $v_k = \Phi(P_k)$. If these n^p real numbers are arranged in increasing size, two possibilities arise. Suppose that at least a third of these values coincide, all being equal to a number v. In this case, $S = \Phi^{-1}(v)$ contains a δ-dispersed subset of size $n^p/3$, and $M(\Phi, \delta) \geqslant n^p/3 > K(\delta)$, as given by (7). Suppose now that fewer than a third of the values v_k are coincident; then we can choose a real number v, distinct from all the v_k but such that at least a third of them are larger than v and a third are smaller. We will show that $S = \Phi^{-1}(v)$ obeys $N(S, \delta) \geqslant K(\delta)$.

Color the lattice point P_k "red" if $\Phi(P_k) = v_k > v$ and "blue" if $\Phi(P_k) < v$. In the p-cell X, a line segment will be called an "RB edge" if it is parallel to an axis and its end points have different colors. Observe that any RB edge must intersect S, and that any two disjoint RB edges are everywhere δ apart; thus, if we can find N mutually disjoint RB edges in X, we will have a δ-dispersed subset of S of size N.

LEMMA. *Color the n^p regular lattice points of the p-cell red or blue in such a way that at least βn^p are of each color; $0 < \beta < 1/2$. Then, the number of disjoint RB edges is at least*

$$2\{\beta n 2^{-p/2}\}^{p-1}. \tag{8}$$

Proof. If $p = 2$, then the n^2 lattice points in the unit square form n rows, each of which is either solid red, solid blue, or mixed. Every mixed row contains an RB edge so that if there are at least βn mixed rows, the lemma holds. Suppose instead that there are fewer than βn mixed rows. The remaining rows cannot all be solid blue for then there would be less than $(n)(\beta n) = \beta n^2$ red vertices in the square, contradicting the hypothesis. We conclude that the square must then have at least one solid red row and one solid blue row, and joining corresponding vertices, we obtain n RB edges.

Suppose now that the lemma has been proved for p cells, and consider a colored $(p+1)$-cell having at least βn^{p+1} vertices of each color. These lie in n parallel sheets, each a p-cell. Call a sheet (mostly) red if it contains fewer than $\beta n^p/2$ blue vertices; a blue sheet is the dual. All other sheets are called mixed. Suppose first that there are fewer than $\beta n/2$ mixed sheets in the $(p+1)$-cell. If there were no red sheets, then the total number of red vertices would be less than

$$(\beta n/2)(1 - \beta/2)\, n^p + (n - \beta n/2)(\beta n^p/2) < \beta n^{p+1},$$

which contradicts the hypotheses. Arguing symmetrically, there must exist at least one red sheet and at least one blue sheet. Matching these sheets, there must be at least

$$n^p - 2(\beta/2)\, n^p = (1 - \beta)\, n^p$$

red vertices that lie above or below a corresponding blue vertex. Joining these, orthogonally to the sheets, we produce $(1 - \beta)\, n^p$ disjoint RB edges, and since this number exceeds what is required by (8), the lemma holds.

The other alternative is that there are at least $\beta n/2$ mixed sheets in the $(p+1)$-cell. We apply the lemma to each, observing that β has now been replaced by $\beta/2$. Accordingly, each sheet contains at least

$$2\{(\beta/2)\, n 2^{-p/2}\}^{p-1}$$

disjoint RB edges. Among all the $\beta n/2$ mixed sheets, there will be

$$\frac{2(\beta n/2)^p}{2^{p(p-1)/2}} = 2\{\beta n 2^{-(p+1)/2}\}^p$$

mutually disjoint RB edges, thus proving the lemma.

To complete the proof of Theorem 2, we apply the lemma with $\beta = 1/3$.

For the mapping class $C[I^p, R^k]$, with $p > k$, the heuristic argument suggests that a correct dispersion function ought to be of the form $K(\delta) \approx C\delta^{-(p-k)}$. However, we have not been able to obtain this, except as shown above when $k = 1$. For the record, we record the following incomplete result which is easily established by a homotopy argument.

THEOREM 3. *If* $\Phi \in C[I^p, R^2]$, *with* $p \geqslant 3$, *then* $M(\Phi, 1) \geqslant 2$.

4. THE CLOSURE OF $\mathscr{F}_Y$

We return to the general problems discussed in the Introduction dealing with the size of the class of factorable mappings between X and Z. Let X, Y, and Z be metric spaces with X compact, and Y simpler than either X or Z. Let $\mathscr{F}_Y$ be the class of mappings F from X into Z that can be factored through Y, as shown below:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle f} \\ & & Z \end{array}, \qquad F = f \circ \Phi. \tag{9}$$

In this section, we examine the situation in which Φ is required to be continuous, while f is unrestricted. We wish to find properties of the class $\mathscr{F}_Y$ which show that only a very restricted subclass of $C[X, Z]$ can be uniformly approximated by the mappings in $\mathscr{F}_Y$. In the next section, we reverse the hypotheses, allowing Φ to be unrestricted but requiring that the functions f obey a uniform Lipschitz condition.

THEOREM 4. *Let* $\{F_k\}$ *be a sequence of mappings from* X *into* Z, *converging uniformly to a continuous mapping* g. *Then, for any* $\delta > 0$,

$$M(g, \delta) \geqslant \limsup_{k \to \infty} M(F_k, \delta). \tag{10}$$

Proof. Since $M(F, \delta) \leqslant N(X, \delta)$ for any $F: X \to Z$, the right side of (10) is an integer N, and there is a subsequence with $M(F_{k_n}, \delta) = N$ for all n. Let $\Delta \subset X^N$ be the compact set of $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ such that $|x_i - x_j| \geqslant \delta$ for

$i \neq j$. Define a continuous function G on Δ by $G(\mathbf{x}) = \max_{i,j} |g(x_i) - g(x_j)|$. Suppose that F belongs to the subsequence $\{F_{k_n}\}$; since $M(F, \delta) = N$, we can choose a point $\mathbf{x} \in \Delta$ and $z \in Z$ such that $F(x_i) = z$ for all $i = 1, 2, \ldots, N$. Then, for any i and j,

$$\begin{aligned} |g(x_i) - g(x_j)| &\leqslant |g(x_i) - z| + |z - g(x_j)| \\ &\leqslant |g(x_i) - F(x_i)| + |F(x_j) - g(x_j)| \end{aligned}$$

and

$$G(\mathbf{x}) \leqslant 2 \, \| g - F \|.$$

Since G is continuous on Δ and $\{F_n\}$ converges to g, there must exist $\mathbf{x} \in \Delta$ such that $G(\mathbf{x}) = 0$. Accordingly, there must exist points $x_1, x_2, \ldots, x_N$, δ-dispersed, with $g(x_1) = g(x_2) = \cdots = g(x_N)$, showing that $M(g, \delta) \geqslant N$.

Return to the mapping diagram (9), and observe that if $F = f \circ \Phi$, then any point-inverse for Φ is automatically a subset of a point-inverse for F, so that $M(F, \delta) \geqslant M(\Phi, \delta)$.

THEOREM 5. *Suppose that $C[X, Y]$ admits the dispersion function $K(\delta)$. Then, $K(\delta)$ is also a dispersion function for the uniform closure of the set $\mathscr{F}_Y \cap C[X, Z]$. In particular, if g is a continuous mapping from X into Z with*

$$\liminf_{\delta \to 0} \frac{M(g, \delta)}{K(\delta)} < 1 \tag{11}$$

then g cannot be approximated uniformly by mappings $F \in \mathscr{F}_Y$.

Proof. Since $M(\Phi, \delta) \geqslant K(\delta)$ and $M(F, \delta) \geqslant M(\Phi, \delta)$ for any Φ in $C[X, Y]$, $K(\delta)$ is also a dispersion function for $\mathscr{F}_Y$. Applying the lemma, we see that $K(\delta)$ is automatically a dispersion function for the uniform closure of $\mathscr{F}_Y$, and must therefore be a lower bound for $M(g, \delta)$, if g can be uniformly approximated by functions F in $\mathscr{F}_Y$. (We note that the same argument applies to subclasses of $C[X, Y]$; if a function $K(\delta)$ can be shown to be a dispersion function for the functions Φ in a subset $\mathscr{S}$ of $C[X, Y]$, then it is also one for the uniform closure of the class of mappings $F: X \to Z$ which factor through Y by means of some $\Phi \in \mathscr{S}$.)

If we use the information in Theorem 2, we obtain:

COROLLARY 1. *Let $n \geqslant m > 1$ and suppose that g is a continuous mapping from I^n into R^m such that*

$$\liminf_{\delta \to 0} \delta M(g, \delta)^{1/(n-1)} < \frac{1}{3(2^{n/2})}.$$

Then, g cannot be approximated uniformly on I^n by mappings of the form $F(x) = f(\Phi(x))$ where f is an arbitrary function on R to R^m and Φ is a continuous real valued function on I^n.

The argument used in Theorem 4 can also be used to compute explicit lower bounds for the distance from a given function g to the class $\mathscr{F}_Y$, when (11) holds.

COROLLARY 2. *If $M(g, \delta_0) < K(\delta_0) = N$, then*

$$d(g, \mathscr{F}_Y) \geqslant \tfrac{1}{2} \min_{x_1, \ldots, x_N} \max_{i,j} |g(x_i) - g(x_j)|,$$

where $\{x_1, \ldots, x_N\}$ is δ_0 dispersed.

Proof. Since we must have $M(g, \delta_0) \leqslant N - 1$, the function G, introduced in the proof of Theorem 4, does not vanish in the set Δ and therefore has a positive minimum γ. If $F \in \mathscr{F}_Y$ then, for an optimal choice of i and j,

$$\gamma \leqslant |g(x_i) - g(x_j)| \leqslant 2 \, \|g - F\|$$

and $\gamma/2$ is a lower bound for the distance from g to $\mathscr{F}_Y$.

Computation of the number γ depends on the explicit nature of the chosen function g. Homeomorphisms provide trivial illustrations. We have $M(\Phi, 1) \geqslant 2^{[n/2]}$ for any $\Phi \in C[I^n, R]$ and $n \geqslant 2$. If g is the identity map of I^n onto itself, then this argument shows that its distance from $\mathscr{F}_R$ is at least $1/2$. If $n \geqslant 3$, Theorem 3 shows that the same holds for the class $\mathscr{F}_{R^2}$. It does not seem likely that these bounds are sharp. We note that g can be approximated by the mapping $F \in \mathscr{F}_R$ given by $F(x) = (w, w, \ldots, w)$, where $x = (t_1, t_2, \ldots, t_n)$ and $w = n^{-1} \sum_1^n t_j$, so that $\|g - F\| = (1/2)\sqrt{n}$ for n even and $(1/2)\sqrt{n - (1/n)}$ when n is odd.

5. Lipschitz Mappings

We now take $Z = X$ and examine the nature of mappings F of X into itself which can be factored through Y as $F = f \circ \Phi$, where Φ is now unrestricted but f is required to be more than merely continuous.

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ & {\scriptstyle F}\searrow & \downarrow {\scriptstyle f} \\ & & X \end{array} \qquad (12)$$

As before, Y is chosen of lower dimension than X. However, Φ can now be 1-to-1, and if f were unrestricted, every map of X into X could be factored as shown, including the identity map. Of course, if $f(Y) = X$, f must be a dimension increasing map. It is known that there exist continuous maps of a k-cell onto an m-cell for any n and m; the familiar Peano "space filling curve" maps I onto I^2 by a function f that is continuous, and is 1-to-1 on an uncountable subset of I. Accordingly, the identity map of I^2 onto itself can be factored through I as $f \circ \Phi$ with f continuous. However, even a small degree of smoothness for f changes the situation, as the next result shows.

THEOREM 6. *Let X have entropy dimension p, and Y have entropy dimension q, and let f be a mapping from Y into X of Lipschitz class α. Then, $f(Y)$ can have entropy dimension at most q/α, and thus f cannot be onto if $\alpha > q/p$.*

Proof. Let $N(f(Y), \delta) = N$ and choose $y_i \in Y$ so that the points $x_i = f(y_i)$ form a δ-dispersed set of N points in $f(Y)$. If $i \neq j$, then

$$\delta \leqslant |x_i - x_j| = |f(y_i) - f(y_j)|$$
$$\leqslant B\, |y_i - y_j|^\alpha$$

and $|y_i - y_j| \geqslant (\delta/B)^{1/\alpha}$. Accordingly, the y_k form a $(\delta/B)^{1/\alpha}$ dispersed set of N points in Y. Since Y has entropy dimension q, $N(Y, \varepsilon) \approx C\varepsilon^{-q}$, and therefore

$$N(f(Y), \delta) = N \leqslant \frac{cB^{q/\alpha}}{\delta^{q/\alpha}}$$

showing that $f(Y)$ has entropy dimension at most q/α.

We conjecture that this result is the best possible, and that there are mappings from I^n onto I^m which belong to Lip(n/m), for every $n < m$; indeed, it is easily seen that the Polya example of a Peano map from $[0, 1]$ onto I^2 is in Lip 1/2, as required.

Let $\mathscr{F}_Y^*(\alpha)$ be the class of mappings F from X into X which factor through Y as $f \circ \Phi$, with f in Lip α, but Φ unrestricted. Since any such map F obeys $F(X) \subset f(Y)$, we have an immediate corollary:

COROLLARY. *If X and Y have entropy dimensions p and q, respectively, with $p > q$, and if $\alpha > q/p$, then $\mathscr{F}_Y^*(\alpha)$ does not contain $C[X, X]$; indeed, no member of it can obey $F(X) = X$.*

To obtain a corresponding result for uniform approximation, we must make a slight change; let $\mathscr{F}_Y^*(\alpha, B)$ be those $F = f \circ \Phi$, where $f \in$ Lip(α, B), with a fixed Lipschitz constant B for all f.

THEOREM 7. *Let X and Y have entropy dimensions p and q, with $p > q$, and suppose that $\alpha > q/p$. Then, $\mathscr{F}_Y^*(\alpha, B)$ is not uniformly dense in $C[X, X]$;*

every mapping g in $C[X, X]$ *that can be uniformly approximated by the class* $\mathscr{F}_Y^*(\alpha, B)$ *must fail to be onto, since* $g(X)$ *will have entropy dimension smaller than p.*

Proof. Given $\delta > 0$, choose a set of N δ-dispersed points in $g(X)$, where $N = N(g(X), \delta)$. Suppose that there is $F \in \mathscr{F}_Y^*(\alpha, B)$ with $\| g - F \| < \delta/3$. Let $z_i = g(x_i)$, and set $y_i = \Phi(x_i)$; then, if $i \neq j$,

$$\begin{aligned}\delta \leqslant |z_i - z_j| &\leqslant |g(x_i) - F(x_i)| + |F(x_i) - F(x_j)| \\ &\quad + |F(x_j) - g(x_j)| \\ &\leqslant 2\, \| g - F \| + |f(y_i) - f(y_j)|\end{aligned}$$

and

$$\frac{\delta}{3} \leqslant B\, |y_i - y_j|^{\alpha}.$$

Accordingly, the y_i form a set of N points of Y that are $(\delta/(3B))^{1/\alpha}$ dispersed, and

$$N(g(X), \delta) = N \leqslant N(Y, (\delta/3B)^{1/\alpha}) \leqslant \frac{C(3B)^{q/\alpha}}{\delta^{q/\alpha}}.$$

Since C, B, and q are independent of δ, this shows that $g(X)$ has entropy dimension at most q/α, which by hypothesis is smaller than p, the dimension of X.

There are many obvious remaining questions about the size and nature of the sets $\mathscr{F}_Y^*(\alpha, B)$ and their relationship to the entire space $C[X, X]$. Furthermore, the complexity measure $N(S, \delta)$ and those derived from it are not the only ones of interest in this context. It would also be interesting to examine these questions in a category different from that of spaces and continuous mappings.

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